Expanding the Area of Gravitational Entropy

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Abstract

I describe how gravitational entropy is intimately connected with the concept of gravitational heat, expressed as the difference between the total and free energies of a given gravitational system. From this perspective one can compute these thermodyanmic quantities in settings that go considerably beyond Bekenstein's original insight that the area of a black hole event horizon can be identified with thermodynamic entropy. The settings include the outsides of cosmological horizons and spacetimes with NUT charge. However the interpretation of gravitational entropy in these broader contexts remains to be understood.

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1 Introduction

It would not be an exaggeration to say that the two major achievements of 20th century physics – quantum mechanics and general relativity – have left us with some of science's most baffling conundrums. From the role of time to the nature of measurement to the definition of a particle, their conceptual foundations are markedly different and are apparently at odds with one another. The spectacular empirical success enjoyed by each of these two pillars of theoretical physics has only served to deepen the conundrums, in large part because there is very little common territory shared by each.

It was a key achievement of Jacob Bekenstein to pioneer such common territory. In proposing that the area of a black hole was akin to thermodynamic entropy [1], he opened up the new subfield of gravitational thermodynamics, in which quantum mechanics and general relativity can simultaneously be put to the test. Although up until now this has been possible only via gedanken experiments, a small but growing body of researchers are seriously entertaining the possibility of creating black holes in a laboratory setting [2]. I shall not explore this tantalizing yet highly speculative idea further here, but the fact that it is being considered at all shows how far the subject has come in the three decades since Bekenstein's original paper.

Once Hawking established that black holes could radiate a thermal spectrum of particles via quantum-field-theoretic effects [3], the relationship between entropy and area that Bekenstein pointed out became a pillar of gravitational thermodynamics. Since the entropy/area relationship does not exist unless $\hbar$ is nonzero, it became a key pivot point in the development of a quantum theory of gravity. Although progress was somewhat slow at first, an enormous amount of effort has been expended in the last decade by the theoretical physics community in understanding the origin of black hole entropy. This research was carried out along two main lines. One involved an investigation of the breadth of situations in which the first law of gravitational thermodynamics applies, including other dimensionalities, dilaton gravity, cosmological settings, higher-derivative theories and string theory. The other concentrated on finding a statistical-mechanical interpretation of black hole entropy by searching for the underlying degrees of freedom in a candidate quantum theory of gravity such as loop quantum gravity or string theory. Many important results on gravitational entropy have been achieved, including its quasilocal formulation [4], its Noether-charge interpretation [5], obtaining and controlling its quantum corrections [6], solvability of toy quantum gravity models [7], its relationship to pair-production of black holes [8, 9], and a partial understanding of its underlying degrees of freedom in terms of string modes [10], spin-networks [11], and boundary diffeomorphisms [12].

Rather than review each of these developments here, my purpose in this paper is to take a small step along the path Bekenstein forged by proposing that gravitational entropy must be present whenever gravitational heat can be defined. The 'heat' referred to here is that given by the Gibbs-Duhem relation, which expresses the heat for any thermodynamic system in terms of the difference between its total energy and its Helmholtz free energy. This relationship was first explored in the context of black holes by Gibbons and Hawking [13], who argued that the Euclidean gravitational action is equal to the grand canonical free energy times the reciprocal of the temperature associated with a black hole event horizon. It is natural to more generally claim that whenever the difference between the total and free energies is nonzero in a given gravitational

setting there will be heat, that form of energy which is useless for doing work. Provided that an equilibrium temperature can be defined (at least approximately), there will then be a gravitational entropy, given by the ratio of the gravitational heat energy to this temperature.

2   Gravitational Heat

Any thermodynamic system at some constant temperature $T = \beta^{-1}$ will have a total internal energy U that can be meaningfully partitioned into two parts: the amount available to do work on some other system (the free energy or Helmholtz potential F), and the remainder, referred to as the heat. This relationship is expressed as

$$U = F + \beta^{-1}S \tag{1}$$

where S is referred to as the entropy of the system. From this perspective, entropy can be regarded as being defined by the ratio of the difference between the total and free energies of the system with its temperature. Alternatively, by taking differentials of eq. (1), the relation $S = \beta^2 \frac{\partial F}{\partial \beta}$ is easily obtained.

Other thermodynamic potentials exist of course – the Gibbs potential, the enthalpy, etc. However the advantage of the Helmholtz free energy is that it admits a straightforward generalization to gravitation via the (Euclidean) path-integral formalism [13]. In other words, one can define gravitational heat to be the difference between the total and free energies of a given gravitational system, with the entropy being the heat divided by the equilibrium temperature.

The logic proceeds in the following manner. For any given system the partition function is

$$Z = Tr\left[e^{-\beta H}\right] \tag{2}$$

where H is the Hamiltonian of the system and the trace is over all of its possible states. Since $U = \langle H \rangle = Tr\left[He^{-\beta H}\right] = -\frac{\partial \ln Z}{\partial \beta}$, the relation

$$\frac{\partial}{\partial \beta}\left(\beta^{-1} \ln Z\right) = -\beta^{-1}\left(\beta^{-1} \ln Z + U\right) \tag{3}$$

is easily seen to hold, from which the identification

$$\ln Z = -\beta F = S - \beta U \tag{4}$$

or

$$S = \beta\left(U - F\right) \tag{5}$$

straightforwardly follows.

To formulate this relation in a gravitational context, one proceeds in an analogous manner. Consider first the action that gives rise to the Einstein equations of motion in $d+1$ dimensions, which is

$$I = I_B + I_{\partial B} \tag{6}$$

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where

$$I_B = \frac{1}{16\pi G} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \left(R - 2\Lambda + \mathcal{L}_M\left(\Psi\right)\right) \tag{7}$$

$$I_{\partial B} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma}\, \Theta \tag{8}$$

$\mathcal{L}_M$ is the Lagrangian for matter fields Ψ, and a cosmological constant has been included. The first term in (7) is the bulk action over the $d+1$ dimensional Manifold $\mathcal{M}$ with metric g . The second term (8) is a surface term necessary to ensure that the Euler-Lagrange variation is well-defined, i.e. that one can fix variations of the metric on the boundary $\partial\mathcal{M}$ without constraining variations of metric derivatives. This boundary (with induced metric γ and extrinsic curvature $\Theta_{\mu\nu}$) in general consists of both spacelike and timelike hypersurfaces, and can either be a boundary of the entire manifold or some submanifold. For example, if $\partial\mathcal{M}$ is a boundary of the entire manifold, it will be the Einstein cylinder at infinity in an asymptotically anti de Sitter (AdS) spacetime, whereas for an asymptotically de Sitter (dS) spacetime it will be the union of spatial Euclidean boundaries at early and late times.

One can then construct the path integral

$$Z = \int D\left[g\right] D\left[\Psi\right] e^{-I[g,\Psi]/\hbar} \tag{9}$$

by integrating over all metrics and matter fields between some given initial and final hypersurfaces. The integration here is the analog of taking the trace in eq. (2). However it is considerably more problematic: there are continuously infinitely many functional degrees of freedom to integrate over, there are redundant gauge and diffeomorphism degrees of freedom that must be eliminated, and the signature of the metric implies that the integration will not converge.

Remarkably enough one can overcome these difficulties, at least in a restricted context. If the spacetime background were flat, the functional integration would only be over the matter degrees of freedom. By analytically continuing the time coordinate $t \to i\tau$, the path integral then formally converges. The formalism of finite temperature quantum field theory [14] then indicates it will become fully analogous to the expression (2) provided that the integration over τ in the action is periodic with some period β. Generalizing to the case of curved spacetimes, a similar set of manipulations can be performed, provided the class of metrics integrated over is stationary. Although the general form of the functional integration is still rather delicate to perform, to leading order in $\hbar$ one easily obtains

$$Z \simeq \mathcal{N} e^{-I_{\mathrm{cl}}/\hbar}$$

where I_{cl} is the classical action evaluated on the equations of motion of the gravity/matter system. Unlike flat-space finite-temperature field theory, however, the period β is no longer arbitrary since curved Euclideanized manifolds with arbitrary periodic time will typically have singularities (conical or otherwise) , which in turn imply the existence of additional matter sources. By demanding regularity of the Euclideanized manifold, such singularities are avoided, and this in turn is accomplished by restricting the period β.

The physical interpretation of the preceding formalism is that the class of regular stationary metrics forms an ensemble of thermodynamic systems at equilibrium temperature β. Application of eqs. (4,5) then yields

$$S = \beta \mathfrak{U} - I_{\mathrm{cl}} \tag{10}$$

setting $\hbar = 1$ and ignoring the normalization factor $\mathcal{N}$, which is irrelevant in this semi-classical approximation. The gravitational entropy S is simply the product of the inverse temperature β with the difference between the total energy $\mathfrak{U}$ and the free energy $\mathfrak{F} = I_{\mathrm{cl}}/\beta$.

3 Conserved Charges

The proposal, then, is that gravitational entropy must be present when there is a difference between $\mathfrak{U}$ and $\mathfrak{F}$ as defined in equation (10) above. Of course if there is no mismatch between $\mathfrak{U}$ and $\mathfrak{F}$ there will be no gravitational entropy. In this case the Euclidean manifold will be regular and the period β arbitrary, as with the standard situation in finite-temperature field theory.

Conversely, if there is a difference between $\mathfrak{U}$ and $\mathfrak{F}$ then there must be gravitational heat, and therefore gravitational entropy. Although the original motivation for obtaining eq. (10) was to understand black holes as thermodynamic systems, an extension to cosmological horizons naturally follows from the formalism [13]. Indeed, as long as the quantities quantities $\mathfrak{U}$, $\mathfrak{F}$ and β can be computed independently there is no *a-priori* reason why eq. (10) should not hold in any context.

One situation in which mismatches between $\mathfrak{U}$ and $\mathfrak{F}$ can occur is when there is a degeneracy in foliating the manifold with foliation parameter τ. Such degeneracies can take place if there is a $U(1)$ isometry (generated by a Killing vector ξ) with a fixed-point set. The existence of any fixed point set makes it impossible to everywhere foliate the spacetime with surfaces of constant τ, leading to a difference between the total energy $\mathfrak{U}$ and the free energy $\mathfrak{F}$ of the gravitational system [15, 16].

Evaluation of $\mathfrak{F}$ is tantamount to evaluation of I_{cl}, and is straightforward for any solution of the Einstein equations. Evaluation of β can be carried out as outlined above, by demanding that the Euclidean manifold remain regular at all fixed point sets of the foliation degeneracy. These fixed point sets will have a co-dimension $d_f \leq d - 1$. If equality does not hold ($d_f < d - 1$) the fixed-point set is called a "nut" [17], in contrast to the "bolt" fixed point sets, whose co-dimension $d_f^{\mathrm{bolt}} = d - 1$. The regularity requirement determines β in terms of the other parameters in the metric; if there is more than one fixed-point set then this imposes additional constraints on these parameters. The physical interpretation is that thermal equilibrium can only be maintained for the set of parameters obeying these constraints (eg. a black hole in thermal equilibrium with the cosmological de Sitter temperature).

Evaluation of the total energy $\mathfrak{U}$ is a more delicate matter. Consider the variation of the action with respect to the metric degrees of freedom $g_{\mu\nu}$. This will produce two types of terms: an integral over $\mathcal{M}$ proportional to the equations of motion, and an integral over $\partial\mathcal{M}$ whose integrand T_{ab}^{eff} is given by the variation of the action at the boundary with respect to γ^{ab}. It is often referred to as

a "boundary stress-energy", though it is not the stress-energy that appears on the right-hand side of the Einstein equations. If the boundary geometry has an isometry generated by a Killing vector ξ, then it is straightforward to show that $T_{ab}^{\text{eff}}\,\xi^b$ is divergenceless with respect to the covariant derivative of the boundary metric. Writing the boundary metric in the form

$$\gamma_{ab}d\hat{x}^a d\hat{x}^b = d\hat{s}^2 = N_t^2 d\tau^2 + \sigma_{ab}\left(d\varphi^a + N^a d\tau\right)\left(d\varphi^b + N^b d\tau\right) \tag{11}$$

where the φ^a are coordinates describing closed surfaces Σ (distinguished by the foliation parameter τ), and N_t and N^a are the lapse function and shift vector respectively. It is straightforward to show that the quantity

$$\mathfrak{Q}_\xi = \oint_\Sigma d^{d-1}S^a T_{ab}^{\text{eff}}\xi^b = \oint_\Sigma d^{d-1}\varphi\sqrt{\sigma}\,u^a T_{ab}^{\text{eff}}\xi^b \tag{12}$$

is conserved between surfaces of constant τ, whose unit normal is given by u^a. Physically this means that a collection of observers on the hypersurface whose metric is γ_{ab} all observe the same value of $\mathfrak{Q}_\xi$ provided this surface has an isometry generated by ξ^b.

If $\xi = \partial/\partial\tau$ then $\mathfrak{Q}_{\xi=\partial/\partial\tau}$ is identified with the conserved mass/energy $\mathfrak{U}$; if $\xi_a = \partial/\partial\phi^a$ then $\mathfrak{Q}_{\xi_a=\partial/\partial\phi^a}$ is identified with the conserved angular momentum $\mathfrak{J}_a$ provided ϕ is a periodic coordinate associated with Σ.

Computing the gravitational entropy for a given (stationary) gravity/matter system is then apparently straightforward: (a) calculate β by enforcing regularity of its Euclidean metric, (b) compute its total energy $\mathfrak{U}$ using eq. (12), (c) compute its classical action I_{cl} from the solution to the Einstein/matter equations, and (d) insert these into (10) to obtain the entropy S. However the volume of Σ becomes infinitely large for the entire spacetime, and so neither the action (6) nor the conserved charges (12) are guaranteed to be finite when evaluated on a solution of the equations of motion. An obvious response to this situation is to calculate all quantities with respect to some reference spacetime, interpreted as the ground state of the system. This can be done by subtracting a term $I_{\text{ref}}\left[g_{\text{ref}}, \Psi_{\text{ref}}\right]$ from (6) and embedding Σ in the background spacetime; conserved quantities then become differences $\Delta\mathfrak{Q}_\xi = \mathfrak{Q}_\xi - \mathfrak{Q}_\xi^{\text{ref}}$ where the latter term is computed from the reference action. Similarly, one must compute $\Delta I = I - I_{\text{ref}}$, and so the entropy for a given spacetime in this approach is in general not intrinsically defined; rather it is given by $\Delta S = S - S_{\text{ref}}$.

Another procedure that comes to mind is to develop a formalism that does not require the boundary to contain the entire spacetime. Such a formalism was originated by Brown and York [4], using a Hamilton-Jacobi type analysis of the Einstein-Hilbert action. Beginning with a timelike vector field defining a flow of time, and a timelike foliation of a finite region of spacetime the action (6) can be decomposed according to this flow and foliation, yielding natural candidates for conserved charges (such as energy and momentum) that are defined for a region of finite spatial extent, i.e. quasilocally.

The advantage of the quasilocal approach is that one can describe gravitational thermodynamics in regions of finite spatial extent. This is in more natural accord with our empirical experience

of thermodynamics, a central concept of which is that of a system and a reservoir that are separated by a partition. Indeed, all physical systems with which we have had any experience have a finite spatial boundary. However the various thermodynamic variables will depend upon some parameter characterizing the size of the region – the radius, for example, if a spherically symmetric region is chosen as the boundary of the system. As this parameter becomes large, these thermodynamic variables will diverge. One can still obtain results for arbitrarily large regions by embedding the quasilocal boundary of the region into a reference spacetime as described above, and then computing all thermodynamic quantities with respect to the referent. This procedure will not affect the first law of thermodynamics for either finite or infinite regions [18, 19]. However the total energy, the entropy, and all other measurable extensive thermodynamic variables will in general depend upon the referent background spacetime.

At first this might appear to be a relatively benign modification to the formalism. However it suffers from at least three significant difficulties. First, in an actual physical situation, observers must contend with the fact that they can only make measurements in the physical spacetime in which they reside. The reference spacetime is one for which they have no access – more simply, there is no guarantee that the ground state is physically attainable, though it logically exists. Hence there is no means of empirically checking values relative to some referent. One might hope that this could perhaps be done for all practical purposes, but even this is not possible. For example one set of observers might surround a gravitating region and make measurements on a quasilocal surface they define, whilst a second set of observers agrees to employ the same quasilocal surface in making their measurements relative to some reference system. The problem is that the second set of observers have no physical guarantee that the system they are surrounding has the properties required for it to be a referent since it is not possible to gravitationally screen this region from other influences, either within or without. For example, dark matter might exist inside it; since by definition such matter can only be detected gravitationally, there is no way to be certain it is not present unless measurements are made with respect to a third referent, plunging the observers into a situation of infinite regress.

These issues could be avoided, at least conceptually, if the reference spacetime could be regarded purely as a mathematical artifact that all observers agreed to refer their measurements to. Unfortunately – and this is the second difficulty – the choice of reference spacetime is not always unique [20]. In principle there can be several different choices one could make for such a referent, all of which are physically reasonable, but each of which yields different answers for the quantities in question. The third difficulty is that observers can choose quasilocal system boundaries for which the referent does not even exist, since it is not always possible to embed a boundary with a given induced metric into the reference background. This problem is not confined to esoteric examples; it occurs even for the simple case of a Kerr black hole, where it has been shown that the embedding problem forms a serious obstruction towards calculating the subtraction energy [21].

4   Boundary Counterterms

It would be desirable, then, to have a formalism that avoids the use of a reference spacetime entirely, defining all thermodynamic variables with respect to quantities intrinsic to the spacetime.

This would resolve the measurement problem since no referent spacetime exists, as well as the embedding problem, since there is no need to embed. One way of doing this is to introduce additional terms in the action that depend only on curvature invariants that are functionals of the intrinsic boundary geometry. Such terms cannot alter the equations of motion and, since they are divergent, offer the possibility of removing divergences that arise in the action (6) for arbitrarily large spacetime regions. Although there exist infinitely many curvature invariants for a given boundary metric, for dimensional reasons only finitely many of them will be non-zero in the limit the boundary contains the entire spacetime. Consequently one need choose only a finite number of coefficients to cancel the divergences of a given spacetime.

In this approach, then, all observers agree to use an action of the form

$$I = I_B + I_{\partial B} + I_{\text{ct}} (\gamma) \tag{13}$$

where the first two terms are given by (7,8) above and $I_{\text{ct}} (\gamma)$ depends on the intrinsic geometry of the boundary. This latter term acts as kind of "counterterm" that cancels out the divergences in the action and conserved charges from the first two terms. Indeed, this approach [22] was inspired by the conjectured AdS/CFT correspondence [23], which states that the partition function of any field theory on AdS_{d+1} is identified with the generating functional Z_{CFT} of a conformal field theory on its boundary $\partial \mathcal{M}_d$ at infinity [24]

$$\left\langle \exp \left(\int_{\partial \mathcal{M}_d} d^d x \sqrt{g} \mathcal{O} \phi_0 \right) \right\rangle = Z_{CFT}[\phi_0] \equiv Z_{AdS}[\phi_0] = \int_{\phi_o} \mathcal{D}\phi \, e^{-S(\phi)} \tag{14}$$

where ϕ_0 is the finite field defined on the boundary of AdS_{d+1}, the integration is over the field configurations ϕ that approach ϕ_0 when one goes from the bulk of AdS_{d+1} to its boundary, and $\mathcal{O}$ is a quasi-primary conformal operator on $\partial \mathcal{M}_d$. While there is no proof at present of this conjecture, there is considerable circumstantial evidence to support it [25] . For example it has been explicated for a free massive scalar field and a free $U(1)$ gauge theory, as well as having been partially confirmed for an interacting massive scalar, a free massive spinor and interacting scalar-spinor fields, as well as for classical gravity and type-IIB string theory. In all these cases, the exact partition function (14) is given by the exponential of the action evaluated for a classical field configuration which solves the classical equations of motion, and explicit calculations show that the evaluated partition function is equal to the generating functional of some conformal field theory with a quasi-primary operator of a certain conformal weight.

Quantum field theories in general contain counterterms, and so it is natural from the AdS/CFT viewpoint to append the boundary term I_{ct} to the action as in eq. (13). However there remains the issue of uniqueness, since it is logically admissible that the coefficients must be chosen differently for each spacetime under consideration, or alternatively, that there exist two different choices of coefficients for a given spacetime that give rise to different finite results. Remarkably enough neither of these situations arises for asymptotically AdS spacetimes; rather the boundary counterterm action is universal, being composed of a unique linear combination of curvature invariants that cancel the divergences that arise in the limit the boundary contains the full spacetime. This was first observed for the full range of type-D asymptotically AdS spacetimes, including Schwarzchild-AdS, Kerr-AdS, Taub-NUT-AdS, Taub-bolt-AdS, and Taub-bolt-Kerr-AdS [16, 26, 27]; shortly

afterward a straightforward algorithm was constructed for generating it [28]. The procedure involves solving the Einstein equations (written in Gauss-Codacci form) in terms of the extrinsic curvature functional $\Pi_{ab} = \Theta_{ab} - \Theta\gamma_{ab}$ of the boundary $\partial\mathcal{M}$ and its normal derivatives to obtain the divergent parts. The algorithm works because all divergent parts can be expressed in terms of intrinsic boundary data and so do not depend on normal derivatives [29]. Explicitly, one writes the divergent part $\tilde{\Pi}_{ab}$ as a power series in the inverse cosmological constant, then covariantly isolates the entire divergent structure for any given boundary dimension d. Varying the boundary metric under a Weyl transformation, it is straightforward to show that the trace $\tilde{\Pi}$ is proportional to the divergent boundary counterterm Lagrangian.

The actual form of $I_{\text{ct}}(\gamma)$ has only been explicitly computed up to 9 dimensions [34]

$$
\begin{aligned}
I_{\text{ct}} = {} & \int \sqrt{-\gamma} \left[-\frac{d-1}{\ell} - \frac{\ell\Theta(d-3)}{2(d-2)}\mathsf{R} - \frac{\ell^3\Theta(d-5)}{2(d-2)^2(d-4)} \left(\mathsf{R}^{ab}\mathsf{R}_{ab} - \frac{d-1}{4(d-2)}\mathsf{R}^2 \right) \right. \\
& + \frac{\ell^5\Theta(d-7)}{(d-2)^3(d-4)(d-6)} \left(\frac{3d+2}{4(d-1)}\mathsf{R}\mathsf{R}^{ab}\mathsf{R}_{ab} - \frac{d(d+2)}{16(d-1)^2}\mathsf{R}^3 \right. \\
& \left. \left. -2\mathsf{R}^{ab}\mathsf{R}^{cd}\mathsf{R}_{acbd} - \frac{d}{4(d-1)}\nabla_a\mathsf{R}\nabla^a\mathsf{R} + \nabla^c\mathsf{R}^{ab}\nabla_c\mathsf{R}_{ab} \right) \right]
\end{aligned}
\tag{15}
$$

where R denotes the curvature tensor (and its contractions) of the boundary, and the step function $\Theta(x)$ is equal to zero unless $x \geq 0$ in which case it equals unity. The quantity $\ell^2 = -d(d-1)/2\Lambda$, and diverges in the flat-space limit.

It was recently shown that this approach can be generalized to asymptotically de Sitter spacetimes [30, 31]. The relevant boundaries $\partial\mathcal{M}^\pm$ are now spatial Euclidean boundaries at early and late times. The boundary and counterterm actions become

$$
I_{\partial B} = \frac{\beta}{16\pi G} \int_{\partial\mathcal{M}^-}^{\partial\mathcal{M}^+} d^dx\sqrt{\gamma^\pm}\Theta^\pm
\tag{16}
$$

$$
I_{ct} = \frac{1}{16\pi G} \int_{\partial\mathcal{M}^-}^{\partial\mathcal{M}^+} d^dx\sqrt{\gamma^\pm}\mathcal{L}_{ct}^\pm
\tag{17}
$$

It has been shown that the divergences of asymptotically de Sitter spacetimes are independent of the boundary normal, and so depend only on intrinsic boundary data [32]. Using this information a counterterm algorithm similar to that for the AdS case can be constructed. The result is

$$
\begin{aligned}
\mathcal{L}_{ct} = {} & \left(-\frac{d-1}{\ell} + \frac{\ell\Theta(d-3)}{2(d-2)}\mathsf{R} \right) - \frac{\ell^3\Theta(d-5)}{2(d-2)^2(d-4)} \left(\mathsf{R}^{ab}\mathsf{R}_{ab} - \frac{d}{4(d-1)}\mathsf{R}^2 \right) \\
& - \frac{\ell^5\Theta(d-7)}{(d-2)^3(d-4)(d-6)} \left(\frac{3d+2}{4(d-1)}\mathsf{R}\mathsf{R}^{ab}\mathsf{R}_{ab} - \frac{d(d+2)}{16(d-1)^2}\mathsf{R}^3 - 2\mathsf{R}^{ab}\mathsf{R}^{cd}\mathsf{R}_{acbd} \right. \\
& \left. - \frac{d}{4(d-1)}\nabla_a\mathsf{R}\nabla^a\mathsf{R} + \nabla^c\mathsf{R}^{ab}\nabla_c\mathsf{R}_{ab} \right)
\end{aligned}
\tag{18}
$$

Turning next to a consideration of conserved charges, if the boundary geometry has an isometry generated by a Killing vector ξ^μ, then a conserved $\mathfrak{Q}_\xi$ can be defined as in (12). However in the

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asymptotically de Sitter context its physical interpretation is somewhat different: a collection of observers on the hypersurface whose metric is γ_{ab} would all observe the same value of $\mathfrak{Q}$ provided this surface had an isometry generated by ξ^b. Although the charge changes with the cosmological time τ, a collection of observers that defined a surface Σ would find that the value of $\mathfrak{Q}$ they would measure is the same as that of other observers collectively relocated elsewhere on the spacelike surface $\partial\mathcal{M}$. Although this surface does not enclose anything, it does not matter – the conserved charge is associated only with Σ, independently of either the structure or existence of the interior of Σ [33].

5 Examples

Consider an evaluation of the energy of Schwarzschild-AdS spacetime with static slicing, that is,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\sigma_{ij}dx^idx^j, \tag{19}$$

where σ_{ij} is a metric of a unit $(d-1)$-dimensional sphere, plane or (compact) hyperboloid for $k = 1, 0, -1$, respectively, and $f(r) = k - 2m/r^{d-2} + r^2/\ell^2$. The Euclideanized metric is obtained by setting $t \to i\tau$

$$ds^2 = f(r)d\tau^2 + \frac{1}{f(r)}dr^2 + r^2\sigma_{ij}dx^idx^j, \tag{20}$$

upon which it is clear that wherever $f(r)$ vanishes there is a degeneracy in foliating the metric with surfaces of constant τ. The metric will thus have a conical singularity in the (τ, r) section unless τ is appropriately periodically identified. Writing $r = \frac{1}{4}f'_+\varepsilon^2 + r_+$, where $f(r_+) = 0$ and $f'_+ = \frac{df}{dr}\big|_{r=r_+}$, the metric becomes

$$ds^2 = \frac{1}{4}\left(f'_+\varepsilon\right)^2 d\tau^2 + d\varepsilon^2 + r^2\sigma_{ij}dx^idx^j, \tag{21}$$

and so the period must be

$$\beta = \left|\frac{4\pi}{f'_+}\right| = \frac{2\pi\ell^2 r_+^{d-1}}{r_+^d + (d-2)m\ell^2} \tag{22}$$

in order to ensure regularity in the region $r \geq r_+$.

Consider a set of observers that place themselves in a collective position of radial symmetry; this is tantamount to enclosing the system with a surface at constant R. This surface has a spacelike unit normal $n^\alpha = \hat{r}^\alpha = (\partial/\partial r)^\alpha$ and extrinsic curvature

$$\Theta_{tt} = -\frac{f'(R)}{2}\sqrt{f(R)} \qquad \Theta_{ij} = R\sqrt{f(R)}\sigma_{ij} \tag{23}$$

and so

$$\left(\Theta_{\mu\nu} - \Theta\gamma_{\mu\nu}\right)u^\mu\xi^\nu = -(d-1)\frac{f(R)}{R} \tag{24}$$

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is the energy density in the absence of the counterterm contributions. When integrated over the surface as in (12), this quantity diverges as $R \to \infty$. However inclusion of the counterterm contributions from the variation of (15) yields

$$
\begin{aligned}
T_{tt}^{\text{eff}} &= \frac{(d-1) f(R)}{8\pi G_n \ell} \left\{ -\frac{\ell \sqrt{f(R)}}{R} + \left[\Theta (d-1) + \frac{k\ell^2 \Theta (d-3)}{2R^2} \right] \right. \\
&\quad \left. -\frac{k^2 \ell^4 \Theta (d-5)}{8R^4} + \frac{k^3 \ell^6 \Theta (d-7)}{16R^6} + \cdots \right\} \\
&= \frac{(d-1) f(R)}{8\pi G_n \ell} \left[\sqrt{1 + \frac{k\ell^2}{R^2}} \right]_d - \sqrt{1 - \frac{2m\ell^2}{R^d} + \frac{k\ell^2}{R^2}}
\end{aligned}
\tag{25}
$$

explicitly showing all terms up to nine dimensions ($d = 8$).

The notation for the first square root means that it is to be understood as a power series that is truncated so that only d terms are retained. This has some interesting implications for large R. If there were no truncation of the series, then the leading term in (25) $\sim mR^{1-d}$, independent of the value of k. For odd d, this remains true even when the series is truncated. However for even d there is contribution proportional to $k^{d/2} R^{1-d}$ that is not cancelled because of the truncation.

Consequently the total energy is

$$
\begin{aligned}
\mathfrak{U}_{\text{AdS}}^{(d,k)} &= \int_\sigma d^{d-1} x \sqrt{\sigma} T_{\mu\nu}^{\text{eff}} u^\mu \xi^\nu = \frac{(d-1) R^{d-1} V_{d-1} \sqrt{f(R)}}{8\pi G_n \ell} \left[\sqrt{1 + \frac{k\ell^2}{R^2}} \right]_d - \frac{\ell}{R} \sqrt{f(R)} \\
&\longrightarrow V_{d-1} \frac{(d-1)}{8\pi G_d} \left[m + \frac{\Gamma \left(\frac{2p-3}{2} \right) \ell^{2p-4} (-k)^{(p-1)}}{2\sqrt{\pi} \Gamma (p)} \delta_{2p,d} \right]
\end{aligned}
\tag{26}
$$

in the limit $R \to \infty$, where ξ^ν is the timelike Killing vector $\partial/\partial t$ and V_{d-1} is the volume of the surface of the compact $(d-1)$-dimensional space (for $k = 1$, see [34]). If $p = \frac{d}{2}$ is a positive integer, the second term remains. This extra term is interpreted as a Casimir energy in the context of the AdS/CFT correspondence conjecture. Its sign depends on the dimensionality and the value of k. Note that slices with $k = 0$ have $\mathbb{R}^4$ topology and are flat, and the Casimir energy vanishes. This is compatible with the features of quantum field theory in curved spacetime.

The action is straightforwardly evaluated. From (7) the bulk component is

$$
\begin{aligned}
I_B &= -\frac{1}{16\pi G_d} \int_{\mathcal{M}} d^{d+1} x \sqrt{-g} \left(-\frac{d(d+1)}{\ell^2} + \frac{d(d-1)}{\ell^2} \right) \\
&= \frac{d}{8\pi G_d \ell^2} \int_{\mathcal{M}_d} d^d \hat{x} \sqrt{\sigma} \int_{r_+}^R dr \, r^{d-1} \\
&= \frac{\beta V_{d-1}}{8\pi G_d \ell^2} \left(R^d - r_+^d \right)
\end{aligned}
\tag{27}
$$

whereas the boundary action (8) is given by

$$
I_{\partial B} = -\frac{1}{8\pi G_d} \int_{\partial \mathcal{M}} d^d x \sqrt{\gamma} \Theta = -\frac{\beta V_{d-1} R^{d-1}}{8\pi G_d} \left[\frac{f'(R)}{2} + (d-1) \frac{f(R)}{R} \right]
\tag{28}
$$

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and the counterterm action (15) is

$$
\begin{aligned}
I_{\text{ct}} &= \frac{1}{8\pi G_d} \int_{\partial \mathcal{M}} d^d x \sqrt{\gamma} \left[\left(\frac{d-1}{\ell} + \frac{k\ell \Theta (d-3)}{2R^2}(d-1) \right) + \frac{k^2 \ell^3 \Theta (d-5)}{8R^4}(d-1) + \frac{k^3 \ell^5 \Theta (d-7)}{16R^6}(d-1) \right] \\
&= \frac{(d-1)\beta V_{d-1} R^{d-1}}{8\pi G_d \ell} \sqrt{f(R)} \sqrt{1 + \frac{k\ell^2}{R^2}} \Bigg]_d
\end{aligned}
\tag{29}
$$

yielding

$$
I = \frac{(d-1)\beta V_{d-1} R^{d-1}}{8\pi G_d \ell} \left(\sqrt{f(R)} \sqrt{1 + \frac{k\ell^2}{R^2}} \Bigg]_d - \frac{\ell f(R)}{R} \right) - \frac{\beta V_{d-1} \left(r_+^d + m\ell^2 (d-2) \right)}{8\pi G_d \ell^2}
\tag{30}
$$

for the total Euclidean action. This becomes

$$
I = \frac{\beta V_{d-1}}{8\pi G_d} \left[m - \frac{r_+^d}{\ell^2} + (d-1) \frac{\Gamma \left(\frac{2p-3}{2} \right) \ell^{2p-4} (-k)^{(p-1)}}{2\sqrt{\pi}\,\Gamma(p)} \delta_{2p,d} \right]
\tag{31}
$$

in the $R \to \infty$ limit

Finally, using eq. (10) the gravitational entropy is given by

$$
S_d = \frac{\left(r_+^d + (d-2)m\ell^2 \right) \beta V_{d-1}}{8\pi G_d \ell^2} = \frac{r_+^{d-1} V_{d-1}}{4G_d} = \frac{1}{4G} A_{d-1}
\tag{32}
$$

which is the familiar entropy/area relation for the $(d+1)$-dimensional class of Schwarzchild anti de Sitter metrics, with A_{d-1} the area of the event horizon.

The preceding derivation is deceptively simple. Since the counterterm action (15) is universal and unique (at least for asymptotically AdS spacetimes) it would suggest that the results (26,31) are also unique, but such is not the case. Under a coordinate transformation, the metric (19) becomes for $k = -1$

$$
ds^2 = g^2 d\mathfrak{r}^2 + a^2 \omega^{4/(d-2)}(-d\mathfrak{t}^2 + \mathfrak{t}^2 \sigma_{ij} dx^i dx^j),
\tag{33}
$$

with $a(r) = e^{-\mathfrak{r}/\ell}$, $g = 2/\omega - 1$ and $\omega = 1 - m/2[a(\mathfrak{r})\mathfrak{t}]^{d-2}$, and an analogous calculation for a collection of observers situated at constant $\mathfrak{r} = \mathfrak{R}$ yields

$$
\mathfrak{U}_{\text{AdS/CFS}}^{(d,k=-1)} = V_{d-1} \frac{m(d-1)}{4\pi G_d}
\tag{34}
$$

for the total energy and action as $\mathfrak{R} \to \infty$. The total energy is now given by a completely different expression for $d =$ even, indicating a clear dependence of these quantities on the foliation of the spacetime.

As a second example, consider the following metric

$$
ds^2 = -V(r) \left(dt + 2n\cos(\theta)d\phi \right) + \frac{dr^2}{V(r)} + (r^2 + n^2) \left(d\theta^2 + \sin^2(\theta)d\phi^2 \right)
\tag{35}
$$

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which describes an asymptotically anti de Sitter spacetime with a non-zero NUT charge N in four dimensions. Here

$$V(r) = \frac{r^2 - n^2 - 2mr + \ell^{-2}(r^4 + 6n^2r^2 - 3n^4)}{r^2 + n^2} \tag{36}$$

This spacetime has singularities along the $\theta = (0, \pi)$ axes, as is easily seen by computing $(\nabla t)^2$ at these locations. Such "Misner string" singularities are the analog of Dirac string singularities that accompany magnetic monopoles, and can be removed by appropriate periodic identification of the time coordinate t [35].

Here again we see an example of a breakdown in foliating the spacetime by a family of surfaces of constant time. In the Euclidean regime this situation can occur in d-dimensions if the topology of the Euclidean spacetime is not trivial – specifically when the (Euclidean) timelike Killing vector $\xi = \partial/\partial\tau$ that generates the U(1) isometry group has a fixed point set of even co-dimension. If this co-dimension is $(d-1)$ then the usual relationship between area and entropy holds. However if the co-dimension is smaller than this the relationship between area and entropy is generalized as we shall see.

Setting $t \to i\tau$ and $n \to iN$ yields

$$ds^2 = F(r)\left(d\tau + 2N\cos(\theta)d\phi\right) + \frac{dr^2}{F(r)} + (r^2 - N^2)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{37}$$

with

$$F(r) = \frac{r^2 + N^2 - 2mr + \ell^{-2}(r^4 - 6N^2r^2 - 3N^4)}{r^2 - N^2} \tag{38}$$

which describes the Euclidean section of this spacetime. Computing the total Euclidean action from (13) gives

$$I_4 = \frac{\beta}{2\ell^2}\left(\ell^2 m + 3N^2 r_+ - r_+^3\right) \tag{39}$$

where r_+ is a root of $F(r)$ and β is the period of τ, given in four dimensions by

$$\beta = 8\pi N \tag{40}$$

and determined by demanding regularity of the manifold so that the singularities at $\theta = 0, \pi$ are coordinate artifacts.

There is an additional regularity criterion to be satisfied, namely the absence of conical singularities at the roots of the function $F(r)$. The argument is similar to that which gave rise to eq. (22) and yields

$$\beta = \left|\frac{4\pi}{F'(r_+)}\right| = 8\pi N \tag{41}$$

where $F(r = r_+) = 0$; the second equality follows by demanding consistency with eq. (40). There are only two solutions to eq. (41), one where $r_+ = N$ and one where $r_+ = r_b > N$, referred to respectively as the NUT and Bolt solutions. The latter solutions have 2-dimensional fixed point sets, whereas the former have 0-dimensional fixed point sets. When the NUT charge is nonzero,

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the entropy of a given spacetime includes not only the entropies of the 2-dimensional bolts, but also those of the Misner strings, and (as noted by Hawking and Hunter [15]) should contribute to the gravitational entropy.

Solving (41) with $r_+ = r_b > N$ gives two possible solutions

$$r_{b\pm} = \frac{\ell^2 \pm \sqrt{\ell^4 - 48N^2\ell^2 + 144N^4}}{12N} \tag{42}$$

where

$$N \leq \frac{(3\sqrt{2} - \sqrt{6})\ell}{12} = N_{max} \tag{43}$$

so that r_b remains real.

Restricting eq. (39) to the NUT and Bolt cases yields

$$I_{NUT} = \frac{4\pi N^2(\ell^2 - 2N^2)}{\ell^2} \tag{44}$$

$$I_{Bolt} = \frac{-\pi(r_b^4 - \ell^2 r_b^2 + N^2(3N^2 - \ell^2))}{3r_b^2 - 3N^2 + \ell^2} \tag{45}$$

with the conserved total energy being

$$\mathfrak{U}_{\text{NUT}} = \frac{N(\ell^2 - 4N^2)}{\ell^2} \tag{46}$$

$$\mathfrak{U}_{\text{Bolt}} = \frac{r_b^4 + (\ell^2 - 6N^2)r_b^2 + N^2(\ell^2 - 3N^2)}{2\ell^2 r_b} \tag{47}$$

for each case. The entropies then follow from the relation (10)

$$S_{\text{NUT}} = \frac{4\pi N^2(\ell^2 - 6N^2)}{\ell^2} \tag{48}$$

$$S_{\text{Bolt}} = \pi\frac{3r_b^4 + (\ell^2 - 12N^2)r_b^2 + N^2(\ell^2 - 3N^2)}{3r_b^2 - 3N^2 + \ell^2} \tag{49}$$

and are no longer given by the areas of the respective event horizons.

The counterterm prescription affords an intrinsic thermodynamic description of the spacetimes described by (35), but with rather strange results [16, 26, 27]. From eq. (48) we see that the entropy becomes negative for $N > \frac{\ell}{\sqrt{6}}$. A further calculation shows that the specific heat becomes negative for $N < \frac{\ell}{\sqrt{12}}$. Hence there are no thermally stable solutions outside of the range

$$\frac{\ell}{\sqrt{12}} \leq N \leq \frac{\ell}{\sqrt{6}} \tag{50}$$

A similar analysis of the bolt case shows that the upper branch solutions ($r_b = r_{b+}$) are thermally stable, whereas the lower branch solutions ($r_b = r_{b-}$) are thermally unstable.

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A forthcoming analysis [36] of higher-dimensional versions of the spacetimes described by (35) indicates that their thermodynamic behaviour is similar for every dimension $d = 4k$. However for $d = 4k + 2$, it can be shown that nowhere are the entropy and specific heat positive for the same values of N.

Corroborating evidence for the preceding results is given by demonstrating that a Noether-charge interpretation of this more general notion of gravitational entropy exists [37].

As a final example, consider $d + 1$ dimensional Schwarzschild-deSitter spacetime, with metric

$$ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\hat{\Omega}_{d-1}^2 \tag{51}$$

where

$$N(r) = 1 - \frac{2m}{r^{d-2}} - \frac{r^2}{\ell^2} \tag{52}$$

and $d\hat{\Omega}_{d-1}^2$ denotes the metric on the unit sphere S^{d-1}. There is a black hole with event horizon at $r = r_H$ and cosmological horizon at $r = r_C > r_H$ (located at the roots of $N(r)$) provided that

$$0 < m \le m_N = \frac{\ell^{d-2}}{d} \left(\frac{d-2}{d} \right)^{\frac{d-2}{2}} \tag{53}$$

where saturation of the inequality yields a spacetime referred to as the Nariai solution. For $m > m_N$, the metric (51) describes a naked singularity in an asymptotically dS spacetime. Hence regularity requirements yield an upper limit to the mass of the SdS black hole.

Working outside of the cosmological horizon, where $N(r) < 0$, the metric can be rewritten as

$$ds^2 = -f(\tau)d\tau^2 + \frac{dt^2}{f(\tau)} + \tau^2 d\tilde{\Omega}_{d-1}^2 \tag{54}$$

where $r = \tau$ and

$$f(\tau) = \left(\frac{\tau^2}{\ell^2} + \frac{2m}{\tau^{d-2}} - 1 \right)^{-1} \tag{55}$$

where the cosmological horizon is at τ_+, defined to be the largest root of $[f(\tau_+)]^{-1} = 0$. Working in this "upper patch" outside of the cosmological horizon, the bulk action is now

$$
\begin{aligned}
I_B &= \frac{d}{8\pi G \ell^2} \int d^d x \int_{\tau_+}^{\tau} d\tau \sqrt{f}\sqrt{h} = \frac{d}{8\pi G \ell^2} \int dt d^{d-1}\hat{x}\sqrt{\sigma} \int_{\tau_+}^{\tau} d\tau \tau^{d-1} \\
&= \frac{V_{d-1}^t}{8\pi G \ell^2} \left(\tau^d - \tau_+^d \right)
\end{aligned}
\tag{56}
$$

where the integration is from the cosmological horizon out to some fixed τ that will be sent to infinity. Here $V_{d-1}^t = \int dt d^{d-1}\hat{x}\sqrt{\sigma}$, with σ^{ab} the metric on the unit $(d-1)$-sphere.

Including the remaining boundary contributions to the action yields

$$
\begin{aligned}
I &= \frac{V_{d-1}^t \tau^{d-1}}{8\pi G} \left[\frac{1}{\ell^2}(\tau - \frac{\tau_+^d}{\tau^{d-1}}) - \frac{1}{2f}\left(-\frac{f'}{f} + \frac{2(d-1)}{\tau} \right) \right. \\
&\left. - \frac{(d-1)}{\sqrt{f}} \left\{ \left(-\frac{1}{\ell} + \frac{\ell \Theta (d-3)}{2\tau^2} \right) + \frac{\ell^3 \Theta (d-5)}{8\tau^4} + \frac{\ell^5 \Theta (d-7)}{16\tau^6} \right\} \right]
\end{aligned}
\tag{57}
$$

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for the total action in this patch. In the large τ-limit this becomes

$$I = \frac{\beta V_{d-1}}{4\pi G_d} \left[m + \frac{\tau_+^d}{2\ell^2} - (d-1)\frac{\Gamma\left(\frac{2p-1}{2}\right)\ell^{2p-2}}{2\sqrt{\pi}\Gamma\left(p+1\right)}\delta_{2p,d} \right] \qquad (58)$$

The total energy is

$$\mathfrak{U}_{\mathrm{dS}}^{(n)} = V_{d-1}\frac{(d-1)}{4\pi G_n} \left[-m + \frac{\Gamma\left(\frac{2p-1}{2}\right)\ell^{2p-2}}{2\sqrt{\pi}\Gamma\left(p+1\right)}\delta_{2p,d} \right] \qquad (59)$$

For odd values of d, the total energy $\mathfrak{U}$ is negative whereas for even values of d it is positive. As the mass parameter m increases, this positive value decreases, approaching its minimum at the Nariai limit. Setting $m = 0$ gives the total energy of dS spacetime in different dimensionalities.

The volumes V_{d-1} are in general divergent, since the t-coordinate is of infinite range. However since $\partial/\partial t$ is a Killing vector, it is tempting to periodically identify it; analytically continuing $t \to it$, yields a metric of signature $(-2, d-1)$. The section with signature $(-,-)$ (described by the (t,τ) coordinates) must have a periodic identification of the t-coordinate with period

$$\beta_H = \left| \frac{4\pi}{(-N'(r))} \right|_{r=\tau_+} = \left| \frac{-4\pi f'(\tau)}{f^2} \right|_{\tau=\tau_+} \qquad (60)$$

so that there is no conical singularity at $\tau = \tau_+$.

This quantity β_H is the analogue of the Hawking temperature outside of the cosmological horizon. Proceeding further, the entropy, as defined by relation (10) becomes

$$S_d = \frac{\left(\tau_+^d - 2(d-2)m\ell^2\right)\beta_H V_{d-1}}{8\pi G\ell^2} \qquad (61)$$

Remarkably enough these entropies are always positive, since $\tau_+^d > 2(d-2)m\ell^2$ so long as $m < m_N$. For example, for $d = 2$, $\tau_+ = \ell\sqrt{1-2m}$ and $\beta_H = 2\pi\ell^2/\tau_+$, yielding

$$S^{d=2} = \frac{\tau_+ V_1}{4G} = \frac{\pi\ell\sqrt{1-2m}}{2G} \qquad (62)$$

The generalized entropy (61) is a monotonically decreasing function of the mass parameter, and is always less than that of empty dS spacetime. Physically it would suggest that a de Sitter spacetime with a black hole is more "ordered" than pure de Sitter spacetime, suggesting that production of black holes in the early universe is entropically disfavored. This is consistent with results on pair-production of black holes in de Sitter spacetime [9].

6   Conclusions

The Gibbs-Duhem relation (10) suggests that gravitational heat should exist whenever the classical action differs from total energy. Gravitational entropy is then given by the difference of these quantities multiplied by the inverse temperature, provided each can be meaningfully defined .

This idea, while dating back more than 25 years to the work of Gibbons and Hawking [13], is now finding application in a wide variety of unexpected areas, made feasible because of the inclusion of the boundary counterterm action into the quasilocal approach. These settings include the outside of cosmological horizons and spacetimes with nonzero NUT charge, as well as constant curvature black holes [38], advancing the concept of gravitational entropy into territory well beyond that of Bekenstein's original suggestion.

Yet some troublesome questions remain. The notion of gravitational entropy presented here rests upon the identification of the gravitational path integral (9) with the thermodynamic partition function (2). Most physicists take this identfication for granted, despite the fact that its foundations rely far more on analogical reasoning than on mathematical precision. Furthermore, although the counterterm action is universal for both asymptotically de Sitter and anti de Sitter spacetimes, (so that only a finite number of coefficients are available to cancel the divergences of a given spacetime), it is not clear these are sufficient to appropriately cancel all possible divergences [26, 31]. Moreover, the counterterm action yields a definition of total conserved charge that is contingent upon the specific foliation of the spacetime [39]. The interpretation of the various thermodynamic quantities in these new contexts (e.g. outside of cosmological horizons) is less than clear. An underlying statistical mechanical interpretation of these results also remains an open question. And, finally, there are no obvious experiments one might perform in the forseeable future to test such ideas, laboratory creation of black holes notwithstanding [2].

Three decades ago Jacob Bekenstein opened the door into an exciting new world when he pointed out the entropy/area relationship. The next three decades should prove to be equally exciting.

Acknowledgments

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